# Multiple Pure Tone Generation in Aeroengine Fans at Subsonic and Supersonic Relative Tip Speeds

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**It has been generally assumed that the multiple pure tones (MPT's) exist only at supersonic relative tip speeds. However, recent narrowband analysis of fan noise at subsonic relative rotor tip speeds has shown that what was previously assumed to be a purely broadband noise, is, in fact, a broadband noise mixed with MPT's. Previous theories, based on shock wave propagation, need to be extended to explain this and other related phenomena. In this paper, the limitations of the linear acoustic equations are reviewed. These limitations make it essential to use a nonlinear model for the MPT generation problem. It is shown that there are three different MPT generation mechanisms. It is shown further that the characteristics given under the first kind of MPT are the ones which are often quoted as the characteristics of the buzzsaw noise. The second kind involves radial modes and would be manifest at higher supersonic relative tip speeds. The third kind is dependent upon the rotor-stator or rotor-inflow interactions. It can and does exist at subsonic relative tip speeds. An example of an application of the theory to the analysis of narrowband data is provided.**

## I. Introduction

### The Tones and MPT's

It has been customary for some time to subdivide the pressure signatures from fans into broadband and discrete tones. The most common of the discrete tones are observed at the blade passage frequency and its harmonics. Thus, if the fan has $B$ blades and if the fan shaft rotates at $N$ rpm, the blade passage frequency is given by

$$f_B = BN/60 \tag{1}$$

For such a fan it is common to observe radiation at $f_B$, $2f_B$, $3f_B$, etc. Such radiation is often described as "fan tones." This paper is concerned mainly with radiation which is neither broadband nor in the form of these tones. Almost all the other kinds of discrete frequency radiation reported in the literature are, at multiples of the shaft rotation frequency, given by

$$f_S = N/60 \tag{2}$$

Such radiation is often called "multiple pure tones." This paper deals mainly with this radiation. In many of the occurrences, these multiple pure tones (MPT's) have a characteristic subjective quality. The sound produced is very similar to that produced by sounding the letter "z" in a repetitious pattern. Under these conditions, the sound and the phenomenon are known as buzzsaw noise.

There are many circumstances (discussed further) when the MPT's are masked by the broadband noise. Under these conditions they would not produce the characteristic sound. Therefore, the MPT is a more general and appropriate term for the discussion in this paper. However, in this and other papers, the two terms are often used interchangeably.

### Various Theories for Supersonic MPT's

An overabundance of reported data regarding the occurence of MPT's deals with this phenomenon, when the rotor relative tip Mach number goes supersonic. When this occurs, shock waves are generated at the top of the rotor. As explained by Hawkings,[1] for example, small variations in the blade spacings and geometries lead these shocks to be of nonuniform strengths and spacings to begin with. However, as the shocks propagate, due to the nonlinear nature of the propagation, these nonuniformities grow. In terms of the fan spectrum, as explained by Hawkings, there is a shift of the energy from the blade passage and its harmonics ("the tones") to the multiples of shaft order tones ("the buzzsaw" or "MPT's").

Hawkings' model was developed further by several workers to arrive at, for example, the predictions of the effect of blade geometry on the MPT's. However, as Hawkings mentioned,[1] the shock wave theory was essentially one- or two-dimensional and did not account for "any three-dimensional effects in either blade aerodynamics or in the duct propagation." A modification of Hawkings' approach was proposed by Mathews and Nagel.[2] They subdivided a variable area inlet geometry into a series of annular streamtubes. Within each of these tubes a propagation of uniformly spaced shock waves was considered to arrive at some impressive conclusions about the effect of the inlet geometry on the extent of buzzsaw generation. However, as the authors pointed out, they had to assume that no acoustic energy could

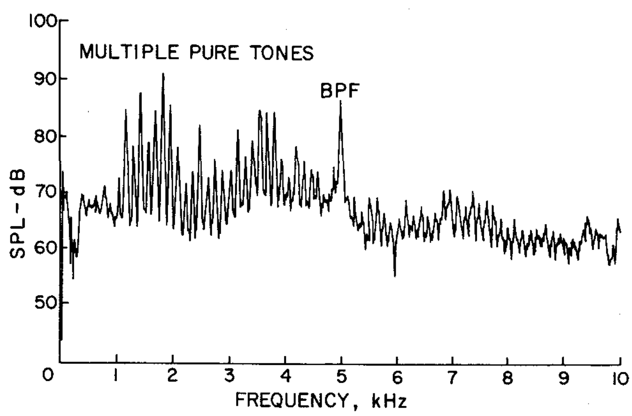


**Fig. 1 Typical spectrum of the multiple pure tones at supersonic relative tip speeds. (From Ref. 13.)**

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flow out of their streamtubes. Therefore, even their work remained, essentially, quasi-two-dimensional.

It was observed in many tests (for example, Ref. 3), that the MPT's were substantially reduced by using a thick inlet lining. It appeared that this lining reduced the MPT's more than the tones (and, in some cases, even increased the tones). Vaidya and Wang[4] reanalyzed Hawkings' model. Hawkings had shown that the nonlinearities of the shock wave propagation lead to the transfer of energy from the tones to the MPT's. Vaidya and Wang used a weak shock analysis developed by Pestorius[5] to model this as a nonlinear resonance phenomenon. They showed that this resonance can occur with and without the presence of shocks. In addition, they showed that the role of a buzzsaw liner (in addition to acting as an absorber of acoustical energy) is to "quench" this nonlinear resonance and to reduce the transfer of blade passage tone energy into its subharmonics. (For a detailed comparison of various buzzsaw theories see Ref. 6.)

#### Experimental Anomalies and New Techniques

This paper is sparked by a large number of experimental observations which were difficult to explain with the help of the unmodified shock wave theories. Perhaps the most conspicuous of these was the presence of the subsonic multiple pure tones. In Sec. II some of these experimental results are reviewed. Also discussed are some of the recent techniques involving narrowband analysis. The modal measurement technique,[7,8] in addition to yielding data about the spectrum, gives information about the spatial distribution of the acoustical energy. Coherence analysis[9] becomes a technique to separate the broadband from discrete tones.

These techniques have confirmed what Saule[10] had shown earlier; that what passes as a broadband noise is often a mixture of a lesser extent of broadband and the MPT's. This conclusion is important from the subjective point of view because the discrete tones at the same decibel level are more annoying than the true broadband. It is also important because the techniques for suppression for the MPT's are likely to be different.

#### Modifications of Previous Theories

In retrospect, the following modifications are necessary to explain some of these experimental discrepancies. First, a modification of the theory to account for three-dimensional acoustical propagation is necessary. In this way, the modification due to the radial modes could be accounted for. More importantly, the earlier theories need to be modified to account for rotor-stator interactions and inflow distortions. To use an analogy, Hawkings' analysis could be regarded as a supersonic (and, therefore, nonlinear) counterpart of Gutin's theory of a single rotor.[11] This paper can be regarded as a nonlinear extension of Tyler and Sofrin's theory.[12]

#### Outline of the Development

In what follows, some of the recent experimental data, suggesting especially the presence of subsonic MPT's, are reviewed. These data serve as a motivation for the development of the subsequent theory, followed by a review of the linear theories of discrete frequency fan noise and their limitations. It has been shown that, especially near cutoff, the linear theories are inadequate. The concept of the secular terms is discussed briefly, followed by a description of the resonance expansion technique to deal with them. It is shown that the linear mechanisms provide the triggers at shaft harmonic frequencies and the blade passage tones provide the source for the growth of MPT's.

This analysis leads to three types of MPT's. The first kind of MPT's is identical to those predicted by the quasi-one-dimensional shock wave theories. The third type, however, is shown to exist even at subsonic relative tip speeds. This is followed by some practical hints about the analysis of narrowband data containing MPT's. As an example, a sample of some of the recent data taken by Joppa is described. It has been shown how the analysis can be used to study such data.

## II. Experimental Evidence

A typical spectrum of the MPT's is shown is Fig. 1, reproduced from Ref. 13. This represents an example of the occurrence of the buzzsaw in the inlet at supersonic tip speeds. Such as occurrence is quite consistent with the leading-edge shock theories. In this section, however, some of the experimental evidence for which the shock theories are inadequate is discussed. A major bulk of engine data is acquired in the form of the third octave spectra. Figure 2, given by Saule,[10] represents such a typical spectrum, obtained for fan B at NASA Lewis. Saule comments that this spectrum is consistent with the notion of discrete tones superimposed on some broadband base.

However, he has shown that the same radiation when analyzed at 20-Hz filter bandwidth shows "several closely spaced noise clusters." A further resolution at 10 Hz was used by Saule to show that these clusters were identified as shaft-order tones. However, even these tones were surrounded by adjacent discrete-tone skirts. At 4 Hz resolution, the shaft-order tones become much more clear. However, at this resolution, several shaft-order tones exhibit sidebands, which are tones formed immediately outside the assigned bands of shaft-order tones. These results were represented by Saule in terms of Fig. 3.

This analysis lends strong support to the case made by Mugridge and Morfey[14] for the use of narrowband analysis. They had estimated that a one-third octave analysis might overestimate broadband noise by as much as 10 dB quite readily.

Saule[10] has supplied further interesting data about MPT's. He has shown far-field, narrowband data for circumferential tip Mach numbers ranging from 0.65 to 0.95. At all of the speeds the MPT's are observed. The relative tip Mach numbers ranged from 0.73 to barely supersonic. Figure 4 shows a part of this data at 40 deg to the inlet axis at a relative tip Mach number of 0.84. The shocks are not supposed to be present at this speed. What is more interesting, in view of the

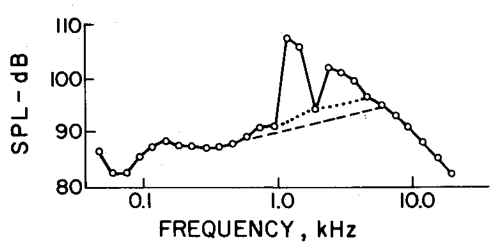


Fig. 2 Third octave spectrum of fan B;.....apparent baseline for broadband,-----true baseline. (From Ref. 10.)

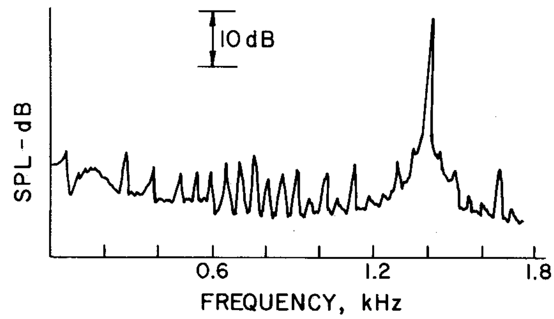


Fig. 3 The same radiation as in Fig. 2, analyzed at 4 Hz handwidth. The actual baseline for broadband is lower. The shaft-order tones and some sidebands at nonintegral shaft orders are seen. (From Ref. 10.)

theory to be developed, is that even at subsonic relative tip speed, in-duct measurements showed, as Saule comments, "a pressure pattern resembling leading-edge shock waves." The same NASA Lewis fan at 140 deg showed MPT's at the subsonic relative tip speeds. This was another example contradicting the predictions of the conventional shock wave theory.

Although Saule's work is quite noteworthy in terms of the data analysis of the NASA fan, his work is by no means isolated in terms of some of these results. In fact he has cited several references describing 1) subsonic tip speed buzzsaw, 2) rear arc buzzsaw, and 3) side bands at nonintegral multiples of the shaft rotational frequencies. Shivashankara[9] has supplemented the technique of narrowband analysis with those of coherence determination and signal enhancement. Broadband is incoherent, whereas the discrete tones are coherent. He has demonstrated that there does exist a significant amount of discrete frequency radiation, which is coherent even at subsonic tip speeds.

Further light on the nature of the buzzsaw is shed through mode measurement techniques. In addition to a knowledge of the frequency spectrum, these techniques yield valuable information about the spatial distribution of the acoustical energy. Recently, Joppa[8] has modified the mode measurement technique developed by Kraft and Posey[7] and others. In collaboration with NASA Langley he used an array of 32 microphones to analyze the data from the inlet of a JT 15D engine. It is shown in a later section that not only the frequency but also the modal content of some of his data is consistent with the predictions of the three types of MPT's (especially the third kind, which is some ways is the most controversial type).

There are other related experimental observations which also serve to motivate this study. The use of inflow control structures to measure fan noise is becoming common these days. There is a controversy about the effect of these structures on buzzsaw. Reference 15, for example, found that there was no effect. Reference 16 found a reduction in buzzsaw on the other hand. It could well be that the two sets of experiments corresponded to different intrinsic mechanisms for the generation of the multiple pure tones.

## III. Discrete Frequency Radiation

### Rotor Alone Field

The radiation of the tone noise was explained by Tyler and Sofrin,[12] using linear acoustics. For a narrow annulus of radius $\bar{a}$, Tyler and Sofrin's theory can be stated as follows: for a rotor with blades, rotating at a speed of $\bar{\Omega}$ rad/s, a pressure field $\bar{p}$ should exist in front of it. If the incoming flow is uniform and the blades are evenly spaced, the field (resulting from the rotor alone) could be represented by

$$\bar{p}=\sum_{\bar{n}} B_{\bar{n}}\cos(\bar{n}B\Psi+a_{\bar{n}}), \qquad (\bar{t}=\bar{z}=0) \tag{3}$$

Where $\bar{t}$ is the time and $\bar{z}$ is the distance along the axis of the duct. Since this field rotates as a whole at the shaft rotational speed $\bar{\Omega}$ rad/s, the dependence on time could be included as

$$\bar{p}=\sum_{n} B_{\bar{n}}\cos(\bar{n}B[\Psi-\Omega t]+a_{\bar{n}}) \tag{4}$$

This is the initial condition. Using this the field upstream of the rotor can be calculated. In the absence of mean flow, if the nonlinear terms are neglected, the wave equation is

$$\frac{1}{\bar{C}_0^2}\frac{\partial^2\bar{p}}{\partial \bar{t}^2}=\frac{\partial^2\bar{p}}{\partial \bar{z}^2}+\frac{1}{a^2}\frac{\partial^2\bar{p}}{\partial\Psi^2} \tag{5}$$

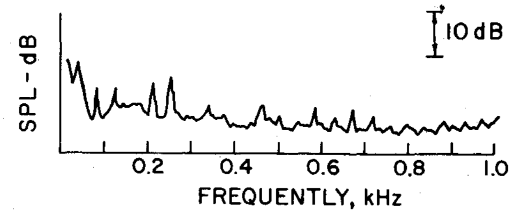


**Fig. 4 Narrowband far-field spectra showing MPT's at the relative tip Mach number of 0.84, and absolute tip Mach number of 0.7, at 40 deg from the inlet axis. (From Ref. 10.) (Even the aft angles show MPT.)**

For outgoing waves, the solution could be assumed to be of the form

$$\bar{p}=\sum_{\bar{n}} B_{\bar{n}}\cos\{\alpha_{\bar{n}}\bar{Z}-\bar{\omega}_{\bar{n}}\bar{t}+\bar{n}B\Psi+a_{\bar{n}}\} \tag{6}$$

The wave equation and the form of the initial conditions give rise to the following relationships:

$$\bar{\omega}_{\bar{n}}=\bar{n}B\bar{\Omega} \tag{7}$$

$$\bar{\alpha}_{\bar{n}}=\frac{\bar{\omega}_{\bar{n}}^2}{\bar{C}_0^2}-\frac{\bar{n}^2B^2}{\bar{a}^2} \tag{8}$$

The solution can be interpreted as a set of spiraling waves since the dependence on time is through $(\psi-\bar{\Omega}\bar{t})$. It is also seen that when $\bar{\omega}_n/\bar{C}_0<nB/\bar{a}$, $\bar{\alpha}_{\bar{n}}$ becomes imaginary. In this case, the wave decays in amplitude as it propagates. In terms of $\bar{\Omega}$, this condition occurs when

$$\bar{\Omega}\bar{a}/\bar{C}_0<1 \tag{9}$$

i.e., when tip Mach number is less than unity. It is interesting and relevant to the problem in the next section that the cutoff frequency of all the modes is reached simultaneously when the rotor tip speed becomes supersonic.

### Rotor-Stator Interaction

Tyler and Sofrin[12] went on to explain the effect of $V$ stator vanes in conjunction with $B$ rotor blades. They showed that the radiation is still at multiples of the blade passage frequency, so that

$$\bar{\omega}=kB\bar{\Omega} \tag{10}$$

but $m$ is given by

$$m=kB\pm sV \tag{11}$$

where $s$ is also an integer.

### The Effect of Blade to Blade Nonuniformities

Mather et al.[17] have shown that if we were to allow small variations from blade to blade, we would get

$$m=\ell\pm sV \tag{12}$$

and

$$\bar{\omega}=\Omega\bar{\ell} \tag{13}$$

Although the radiation through this linear mechanism is expected to be small, it serves as a "trigger" for the nonlinear mechanisms proposed. In the context of the shock wave explanation also, the blade to blade variations play the same role.

**The Effect of Radial Modes**

In a hollow circular duct or in a cylindrical annulus, the radial dependence has been shown by Tyler and Sofrin to be

$$R(r) = G_{m,n}[\mu_{m,n}(r/a)] \tag{14}$$

where $G_{m,n}$ is a combination of the Bessel and Neumann functions. $\mu_{m,n}$ is the eigenvalue, such that $G$ satisfies the boundary conditions. $m$ corresponds to the values discussed above and represents the number of circumferential nodes. $n$ corresponds to the number of radial nodes.

The expression for $\bar{p}$ is given by

$$\bar{p} = \sum_{m,n} A_{m,n} e^{im\Psi} G_{m,n}[\mu_{m,n}(\bar{r}/\bar{a})] e^{i\alpha_{m,n}\bar{k}\bar{z}} \tag{15}$$

where

$$\alpha_{m,n} = \left(1 - \frac{\mu_{m,n}^2}{\bar{k}^2\bar{a}^2}\right)^{1/2} \tag{16}$$

and

$$\bar{k} = \bar{\omega}/\bar{C}_0 \tag{17}$$

**The Effect of Mean Flow**

Morfey[18] has given an extension of these theories to the case when axial and tangential mean flow are considered.

In the case of the axial mean flow of Mach number $M_z$, Eq. (16) is modified to

$$\alpha_{m,n} = \frac{-M_z \pm (M_z^2 + (1-M_z^2)(1-\mu_{m,n}^2/\bar{k}^2\bar{a}^2)^{1/2}}{1-M_z^2} \tag{18}$$

## IV. Cutoff Criteria

Once $\alpha_{m,n}$ in the previous section goes imaginary, the pressure field decays along $z$. It turns out that even for the nonlinear analysis the cutoff phenomenon is important. Just near cutoff the waves spiral around several times before they exit the duct. This makes them more amenable to nonlinear interactions. The various cutoff criteria are often expressed in terms of the cutoff Mach number defined as

$$M_{m,n} = \mu_{m,n}/m \tag{19}$$

In the case of a narrow annulus, the properties of the Bessel functions could be used to show that

$$M_{m,n} \simeq 1.0$$

Also, the expressions of the previous section can be used to show that

$$m = \bar{n}B \tag{20}$$

$$\bar{\omega} = \bar{n}B\bar{\Omega} \tag{21}$$

Since the tip Mach number is

$$M_t = \bar{\Omega}\bar{a}/\bar{C}_0 \tag{22}$$

the condition that $\alpha_{m,n}$ be real or that

$$\bar{k}\bar{a} > \mu_{m,n} \tag{23}$$

translates into

$$M_t > (M_{m,n} = 1) \tag{24}$$

in the absence of the axial mean flow $M_z$.

In the presence of $M_z$, Eq. (18) can be seen to lead to

$$M_{RT}^2 = (M_z^2 + M_t^2) > 1 \tag{25}$$

$M_{RT}$ is the relative tip speed.

Thus the mode is cut on at exactly the same time as the relative tip speed goes supersonic.

Consider now the general case involving stators and the axial mean flow, where

$$m = \ell \pm sV \tag{26}$$

$$\bar{\omega} = \ell\,\bar{\Omega} \tag{27}$$

Therefore, using Eq. (18), it can be readily seen that cut on occurs when

$$M_t > (m/\ell) M_{m,n} \sqrt{1-M_z^2} \tag{28}$$

Now if we define

$$M_{m,n}^* = \frac{m}{\ell} M_{m,n} = \left(\frac{\ell \pm sV}{\ell}\right) M_{m,n} \tag{29}$$

it is clear that if $M^*$ is unity, cut on occurs when $M_{RT}$ is unity. If $M^*$ is less than one, the cut on takes place when $M_{RT}$ is less than unity, and vice versa.

Consider the $m$th mode in $\ell$th harmonic as described above. If such a mode has $M_{m,n}^*$, which is less than one, it could be cut on at subsonic $M_{RT}$. For a far-field detection of a mode, two conditions have to met: first, the mode has to be generated in sufficient quantity; second, it has to be cut on.

A multiple pure tone will be called "subsonic" if it can be present at subsonic relative tip speed. We can see that a *necessary* condition for subsonic MPT is

$$M_{m,n}^* < 1 \tag{30}$$

The condition is not *sufficient* due to the generation condition described above. Linear mechanism will show that at

$$\ell = \bar{n}B \tag{31}$$

that is, at the frequencies given by the blade passage tones, significant generation is possible. Now, if we choose $m < \ell$, (i.e., with negative $S$) it is possible to meet condition (30). Thus we have verified a well-known observation that, due to the rotor-stator interaction (and inflow distortion), subsonic tones are radiated in practice. We have also seen that if subsonic MPT's could be generated they might also propagate for those modes, satisfying condition (30).

We will show in the next section that, given the presence of these modes, however small, the well-generated and well-propagated interaction tones could transfer enough energy into these MPT's to make them experimentally detectable.

## V. Limitations of the Conventional Acoustic Equation

The homogeneous wave equation

$$\bar{C}_0^2 \nabla^2 \bar{\phi} - \frac{\partial^2 \bar{\phi}}{\partial \bar{t}^2} = 0 \tag{32}$$

is often taken to the starting point in many problems in duct acoustics. $\bar{\phi}$ is the velocity potential. Velocity is given by its gradient and pressure, another scalar is, simply, $-\bar{\rho}_0(\partial\bar{\phi}/\partial\bar{t})$. Solutions to this equation are well known from electromagnetism and have helped further the understanding of duct acoustics.

It should be recalled that this equation originated from the Navier-Stokes equations by neglecting some higher order terms. In the absence of viscosity and heat conduction, if those terms are not neglected, we obtain (see, for example, Refs. 19 and 20)

$$C_0^2\nabla^2\bar{\phi}-\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2}=(\gamma-1)\frac{\partial\phi}{\partial\bar{t}}\nabla^2\bar{\phi}+2\nabla\bar{\phi}\cdot\nabla\frac{\partial\bar{\phi}}{\partial\bar{t}}$$
$$+\frac{(\gamma-1)}{2}(\nabla\bar{\phi})^2\nabla^2\bar{\phi}+\frac{1}{2}(\nabla\bar{\phi}\cdot\nabla)(\nabla\bar{\phi})^2 \tag{33}$$

This could be rewritten as

$$C_0^2\nabla^2\bar{\phi}-\frac{\partial^2\bar{\phi}}{\partial\bar{t}^2}=N(\bar{\phi}) \tag{34}$$

where $N(\bar{\phi})$ are the nonlinear terms. It is usually assumed that these terms have a negligible effect on the solution.

Let us look at the problem as involving a simple perturbation. Thus, let us recast Eq. (5) in the following nondimensional form:

$$L(\phi)=\epsilon F(\phi) \tag{35}$$

where $\phi$ is a perturbation parameter. One such parameter could be the acoustic Mach number in the $z$ direction, $V/C_0$. This parameter could well be assumed to be much less than unity. We can then assume that the final solution could be written as

$$\phi=\phi_0+\epsilon\phi_1+\epsilon^2\phi_2+\ldots \tag{36}$$

It is usually hoped that $\phi_0,\phi_1,\phi_2$ would be of the same order. In that case, $\phi=\phi_0$ would be an acceptable solution with an error on the order of $\epsilon$ and $\phi_0+\epsilon\phi_1$ would be a solution correct to the first order, and so on. The procedure to obtain $\phi_0$, $\phi_1$, and $\phi_2$ then could be described by

$$L(\phi_0)=0 \tag{37}$$

$$L(\phi_1)=F(\phi_0) \tag{38}$$

$$L(\phi_2)=[F(\phi_0+\epsilon\phi_1)-F(\phi_0)]/\epsilon \tag{39}$$

and so on.

A problem would be encountered if the right-hand sides of Eqs. (38), (39),..., etc., contain terms which are in resonance with the operator, $L$. Such terms are called the secular terms. In such a case, the ordering implied by Eq. (37) is not valid. To avoid this problem, a method was developed called the resonance expansion.[21] This method bunches together a set of modes of Eq. (39) by means of undetermined coefficients. Following the technique of multiple scales,[19] these coefficients are also made to be functions of the axial distance along the duct. The unknowns are evaluated by using the condition that the secular terms should become identically equal to zero.

## VI. Results from Nonlinear Analysis

The details of the techniques of nonlinear analysis are given in Refs. 19-21 (and also in the references cited in this section). We will concern ourselves only with the results as they influence the generation of the MPT's.

### Steepening of Waveforms and Formation of Shock Waves

The most common case found in the literature deals with an initially sinusoidal wave (see, for example, Ref. 22). It has been shown that the higher the amplitude and the longer the distance the wave travels, greater and greater distortion of its waveform takes place. In the absence of dissipative mechanisms such a wave soon changes into a shock wave. The shocks dissipate thereafter due to viscosity and heat conduction.

Shocks could also form if a blade, for example, goes supersonic in a duct. However, no matter how the shocks are formed, as long as their strength is not very high, they could be analyzed by conventional nonlinear acoustic techniques. This was demonstrated by Pestorius.[5] Through Pestorius' work, the isomorphism between the shock wave theories of MPT and the one described here is well documented.

### The Generation of Subharmonics

One of the key results of the Vaidya-Wang study was that in a narrow annulus the spinning modes show a strong nonlinearity near cutoff. Physically this is due to the high spiral that the higher order modes undergo near the cutoff. Thus, the equivalent length they travel before coming out of even a short duct is quite large. Consider a set of modes, all near cutoff, all going at the same group speeds. Let $A_j$ be their amplitudes. In particular, let their respective phases be 0 or $\pi$, and if we define

$$B_j\equiv jA_j\cos a_j \tag{40}$$

Reference 21 has shown that the following interaction equations result:

$$\frac{dB_1}{dZ}=-B_1B_2+B_2B_3+B_3B_4+\ldots$$
$$\frac{1}{2}\frac{dB_2}{dZ}=\frac{1}{2}B_1^2-B_1B_3+B_2B_4+B_3B_5+\ldots$$
$$\frac{1}{3}\frac{dB_3}{dZ}=B_1B_2-B_1B_4+B_2B_5+B_3B_6+\ldots$$
$$\frac{1}{4}\frac{dB_4}{dZ}=\frac{1}{2}B_2^2+B_1B_3-B_1B_5+B_2B_6+B_3B_7+\ldots \tag{41}$$

The set of equations is generalized Volterra equations. These have been applied to various problems, including those involving the study of turbulence and ecology. In the case of ecology, various $B_j$'s clearly represent the prey and predator roles.[23]

One particular case, which is similar to the prey and predator problem, is that of the generation of subharmonics. Nonlinear interaction will generate multiples of the input frequency. These equations show a mechanism by which energy in the higher harmonics can be depleted by the lower harmonics. Consider, as an example, a problem where the fourth harmonic, say, has an initial amplitude of unity, and all other harmonics are on the order of $\epsilon$ to begin with. In that case, the form of Eq. (41) would be

$$\frac{dB_1}{dZ}=-B_3;\qquad \frac{1}{2}\frac{dB_2}{dZ}=-B_2\ldots\text{etc.} \tag{42}$$

The equations are correct up to first order in $\epsilon$ and valid for small distances from the origin. If truncated, they would be governed by a characteristic determinant. It is seen from this determinant [and, in fact, from the form of Eq. (41)], that certain initial conditions (on phases of the $B$'s) would lead to growth of the subharmonics. This growth can occur up to the point where any of the harmonics breaks through its initially limited order of magnitude. Once that point is reached the problem is nonlinear and Eq. (41) would have to be used.

If the duct walls are not rigid, it has been shown in Ref. 21 that the wall admittance becomes a critical factor. If it is much smaller than the perturbation parameter $\epsilon$, the resonance persists. If it is much greater, the problem becomes linear. If it is on the order of $\epsilon$, the exact equations of Ref. 21 have to be used.

### The Source and the Trigger

These interaction equations show that for the creation of subharmonics, the following criteria have to be met: 1) nonlinear resonance must be present, 2) the modes should travel at close group speeds, 3) the source must be present at a finite amplitude, and 4) the triggers must be present at the subharmonic levels. However, these can be small.

### The Effect of Mean Flow

Vaidya and Wang[24] have shown that, in the presence of a uniform mean flow, the conditions of the previous subsection remain unchanged, except that the region of resonance is broadened for a larger range above cutoff.

The resonance is especially strong for upstream propagation.

## VII. Three Types of MPT's

First, let us consider the case of a narrow annulus. This can be readily shown to be similar to that of a rectangular duct. In this case, the criterion of the previous section generalizes to one as follows.

A mode at azimuthal mode number $m_1$, radial mode number $n_1$, and at frequency $\bar{\omega}_1$, will potentially interact strongly with any other mode at azimuthal mode number $m_2$, radial mode number $n_2$, and circular frequency $\bar{\omega}_2$, such that

$$\frac{m_1}{m_2}=\frac{n_1}{n_2}=\frac{\bar{\omega}_1}{\bar{\omega}_2} \tag{43}$$

It can be readily verified that this condition leads to identical phase and group velocities for the modes. Once condition (43) is satisfied, the actual rate of interaction depends upon[4] the intensity, closeness to cutoff, shape of the duct, mean flow, and shear.

### First Kind

This type of buzzsaw occurs due to rotor field alone. This field consists of various azimuthal and radial $(m,n)$ modes. However, the first kind deals only with $(m,0)$ modes. Since only the azimuthal modes are involved, the criteria of Eq. (43) simplify to

$$\frac{m_1}{m_2}=\frac{\bar{\omega}_1}{\bar{\omega}_2} \tag{44}$$

If there are $B$ blades, most of the energy has a tendency to go into blade passage frequency and its harmonics. The blade passage frequency would be at $B$. Therefore, Eq. (39) would be satisfied by $m_2=1$ at $\bar{\omega}=\bar{\Omega}$, or $m_2=2$ at $\bar{\omega}_2=2\bar{\Omega}$, or $m_2=3$ at $\bar{\omega}_2=3\bar{\Omega}$, etc. Thus, for example, if $B=15$, we can have energy transfer from the blade passage frequency to 1/15th, 2/15th, 3/15th of the blade passage frequency.

Most buzzsaw noise is probably of this type. Validity of this theory is being investigated as good modal measurement techniques are becoming available. Conversely, if it is possible by design to suppress all other kinds of buzzsaw generation, then this principle could be used to investigate and calibrate modal measurement systems.

One strong characteristic of the first kind of buzzsaw follows from Eqs. (9) and (30). In a narrow annulus, every mode of this buzzsaw is cut on simultaneously when the rotor tip speed is just sonic. It turns out that even when inlet axial flow of Mach number $M_z$ is considered, the criterion just changes to that predicted by the shock wave theories,

$$M_{RT}=(M_z^2+M_T^2)^{1/2} \tag{45}$$

### Second Kind

The second kind of buzzsaw also originates in the rotor field. In this case, the radial mode number $n$ is anything other than zero. Now, we have to revert back to Eq. (43). Let us look at $B=15$, and let the blade passage frequency be 15,000 Hz. At this frequency, based on the details of rotor loading, not only will the (15,0) mode be generated, but also (15,1), (15,2), and (15,3) modes will be generated, and so on. In the case of $n\neq 0$, Eq. (43) is sowewhat difficult to satisfy. Table 1 shows the three packets of modes, which will exchange energy among each other.

**Table 1 Resonant packets**

| Packet | $m$ | $n$ | $f$, Hz |
|---|---|---|---|
| A | 15 | 5 | 15,000 |
| A | 3 | 1 | 3,000 |
| B | 15 | 3 | 15,000 |
| B | 5 | 1 | 5,000 |
| C | 15 | 15 | 15,000 |
| C | 15 | 14 | 14,000 |
| C | 13 | 13 | 13,000 |
| C | — | — | — |
| C | — | — | — |
| C | — | — | — |
| C | 1 | 1 | 1,000 |

The higher the $n$, the higher the cut on frequency. As the rotor tip speed reaches sonic from below, none of the packets are cut on. Packet B will be cut on at a supersonic tip speed. Packet A will be cut on at a higher speed yet. Packet C will be cut on at a much higher tip speed. It is likely that very high $n$ numbers carry less and less energy.

Therefore, two surprising results emerge. The first is that the critical relative tip Mach number

$$M_{RT}>1 \tag{46}$$

for the second type of buzzsaw. The second important conclusion is that the higher the lowest common factor for $B$, the higher the critical Mach number is pushed. In particular, if $B$ were to be a prime number, the critical Mach number will be pushed very high and very likely the amplitude of the second kind of MPT will also be greatly reduced.

### Third Kind

The third kind of buzzsaw occurs in the presence of rotor-stator or inflow-rotor interactions. In particular, let us consider the scattering of the rotor field by $V$ stator vanes equally spaced apart.

$$\bar{\omega}=kB\bar{\Omega} \tag{47}$$

but $m$ is given by

$$m=kB\pm sV \tag{48}$$

where $s$ is also an integer.

If we were to allow small variations from blade to blade, we would get

$$m=\ell\pm sV \tag{49}$$

$$\bar{\omega}=\ell\bar{\Omega} \tag{50}$$

Further, variations among the stators might lead to additional variation in $m$. Also since this expression allows for azimuthal orders lower than $B$, cut on below $M_{RT}=1$ is possible.

This type of field can also have $n\neq 0$. In that case, the generation is restricted and the critical Mach number is higher. Despite this, in view of Eq. (30), subsonic critical Mach numbers are possible.

It should be recognized that the influence of inflow distortion also follows Eq. (48). In that case, the most general case is given by $V=1$ and various values of $m$ are possible. In particular, because of the negative sign, $m$ can be quite small and, at subsonic tip speeds, the modes are cut on and the third type of MPT is possible.

## VIII. The Case of a Large Annulus

In the case of a large annulus, Eq. (43) is no longer applicable. We have to resort to the group/phase velocity criteria. As a first step, consider the axial phase velocity. This quantity also has a significant role to play in linear acoustics. In the case of a plane wave propagating in the axial direction, this velocity is the speed of sound. A wave traveling at an angle to the axis will create an axial phase velocity which is supersonic (see Fig. 5). A cut on mode, for this reason, has a supersonic axial phase velocity. At the point of cut off the velocity is infinite, and below cut off it is imaginary.

### First Kind

Figure 6 shows the tip Mach number required for a given $(m,0)$ mode generated by a rotor alone.[25] It can be seen that, for a narrow annulus, for a small $m$ number, the critical Mach number is about the same for all the modes. In the limit for hub-to-tip ratio tending to unity, the critical number for all modes would be unity.

Coincidence of critical Mach number also implies coincidence of axial phase velocity. It is clear, therefore, that for a small number of blades and narrow annulus, the "first kind" of buzzsaw readily occurs. This type of strong interaction can be readily demonstrated in a spinning mode facility.[26]

For lower hub-to-tip ratios, Fig. 6 shows that, for lower spinning orders, the cutoff Mach numbers differ substantially. However, for higher spinning numbers, the effect of the hub-to-tip ratio on the cutoff Mach number is minimal. Also, the difference of $M_{m,0}$ for two neighboring $m$'s is not very large. For the first kind of MPT, in the case of a rotor with 46 blades, even the subharmonics at $m=30$ would have only about 2% difference in the cutoff Mach number. The lower harmonics, however, would have a progressively differing phase velocity and, therefore, would participate in the interaction to a lesser degree.

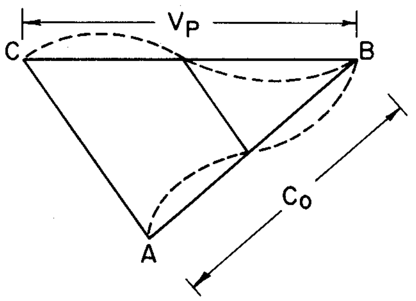


**Fig. 5 Phase velocity $V_p$ for a plane wave going along AB.**

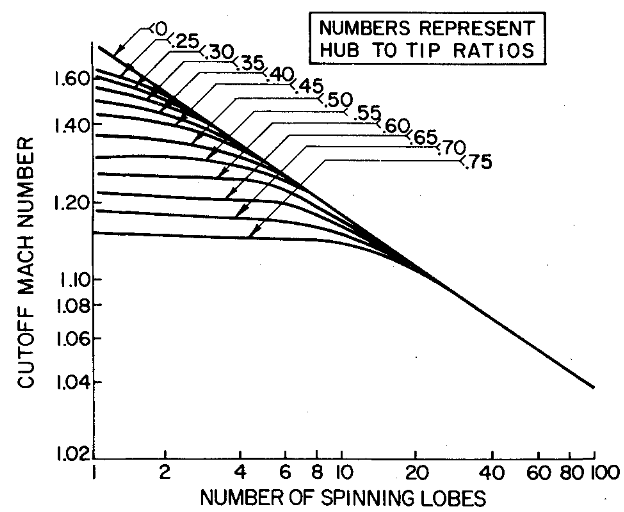


**Fig. 6 Cutoff Mach number for spinning modes. (From Ref. 25.)**

An extreme example of this would occur when the phase velocities of the lower orders go imaginary while those of the higher modes remain supersonic. In this case, the lower order modes would be cut off. This has been the linear version of the explanation of the cut off proposed by Pickett.[27] The phase velocity criterion agrees with this explanation.

It should be noted that even the first kind of buzzsaw begins to occur at a critical relative tip Mach number which is slightly supersonic. The higher the number of blades, the closer to unity this number will be. This could be seen from Fig. 6. However, the asymptotic expression for the roots of the Bessel function could be used to show that

$$M_{m,0} = 1 + 0.8086m^{-2/3} \tag{51}$$

for large $m$. This shows how close $M_{m,0}$ can get to unity for large $m$. This also shows that, for large $m$, $M_{m,0}$ for high subharmonics is not that different from one another.

### Second Kind

The criterion given by Eq. (43) does not hold exactly. However, it can be used as a rough guide to find which $M_{m,n}$'s are likely to be close. In the general case,

$$M_{m,n} = \mu_{m,n}/m \tag{52}$$

where $\mu_{m,n}$ is the $(n+1)$ eigenvalue of the duct mode.

Consider a rotor with 46 blades and a hub-to-tip ratio of zero (hollow cylinder). The blade passage frequency would be generated at $m=46$. Now, in the second radial mode,

$$M_{(46,2)} = 1.324 \tag{53}$$

It turns out that not the 23rd mode, but the 25th mode will interact strongly with this mode, for

$$M_{(25,1)} = 1.328 \tag{54}$$

which is fairly close. Even the (5,0) mode should be generated because

$$M_{(5,0)} = 1.329 \tag{55}$$

If we now consider the second harmonic, there are many potential interactions. For example, (92,4), (70,3), (48,2), and (26,1) have very close critical Mach numbers of about 1.31. At a lower tip Mach number (92,3), (64,2), and (35,7) will be cut on and for all will interact strongly, because of the $M_{(m,n)}$ for all being approximately 1.26. (92,1) has a critical Mach number of 1.13, and at a value that low it would have a phase velocity very close to that of several modes. Therefore, it would be hard to distinguish its impact from that of the first kind of MPT.

There is one more consequence of the inapplicability of Eq. (43). If the number of blades $B$ were to be a prime number, for a narrow annulus the number of wavepackets in resonance would be greatly reduced. This is not necessarily true for the larger annulus. As a result, the advantage of having a prime number of blades is probably reduced in the case of a larger annulus.

### Third Kind

The third kind of MPT is unique in that it can exist even at the subsonic tip speeds. In a large annulus, at fairly subsonic speeds, it would have a tendency to be weak. This is due to the fact that only the lower $m$ order modes will be cut on. However, Fig. 6 shows that, at lower $m$ orders, the critical numbers, $M_{m,0}$'s, for various spinning orders differ quite a bit. Therefore, the resonance in general is weaker. As speeds increase (even while remaining subsonic) the possibility of higher spinning modes satisfying the following criteria increases:

$$M_t \geq (1-M_z^2)^{1/2} M_{m,0}(m/kB) \tag{56}$$

Thus, the higher the $M_t$, the higher the $m$ possible.

At higher speeds especially, the possibility of an inflow distortion causing some energy to go into cut on modes increases. Under these conditions, the interaction MPT might become observable. At supersonic tip speeds, this kind of MPT can exist and might not be readily distinguishable from the first or second kind of MPT.

## IX. Analysis of Narrowband Modal Data

One of the practical applications of the theory of this paper has to do with analyzing narrowband modal data. As a sample case, consider the fan studied by Joppa.[8] It had 28 rotor blades and it was run at 10,988 rmp. The hub-to-tip ratio was 0.405, $M_z$ was 0.26. The tip Mach number was 0.89. Therefore, from Eq. (45), the relative tip Mach number is 0.93. It is clear from the discussion that the only type of MPT's we might expect would be the third kind.

The blade passage frequency for this fan, given by Eq. (1), can be calculated as

$$f_B = \frac{28 \times 10{,}988}{60} = 5128 \text{ Hz} \tag{57}$$

Measurements at this frequency showed a strong tone. There was also a distinct but diminished tone at half the blade passage frequency: 2564 Hz. At both these frequencies Joppa[8] made the modal analysis, using a 32 microphone array. It was found that at the blade passage the dominant mode is $m=22$, and at half the blade passage, the mode is $m=11$.

The presence of the blade passage frequency could readily be explained if one were to account for the six struts present. In Eq. (49), $B$ is equal to 28, and if $V=6$ for the six struts, $s=-1$ gives $m=22$ as observed.

For the $m=22$ mode, in the $\ell=28$th harmonic of the shaft rotational frequency,

$$M^*_{22,0} = \frac{22}{28} M_{22,0} \tag{58}$$

From Fig. 6 (or, more accurately, from the formula)

$$\bar{M}^*_{m,n} \equiv M_{RT\text{crit}} = \{M^{*2}_{22,0} + M_z^2(1-M^*_{22,0})\}^{1/2} = 0.87 \tag{59}$$

Therefore, (22,0) is cut on for relative tip speeds greater than 0.87. This explains that the blade passage was cut on. Now at half the blade passage frequency, $m=11$ meets the resonance criterion (43). Further, $M_{11,0}=1.16$. And $M^*_{11,0}$ for $m=11$ and $\ell=14$ is 0.91.

Thus at the relative tip Mach number of 0.93 both (22,0) at 5128 Hz and (11,0) at 2564 Hz are cut on. Second, they are fairly close to cut on. Third, their phase velocities (as evidenced by $M^*_{m,0}$'s) are fairly close. Fourth, the blade-to-blade variation would provide a small trigger at (11,0), according to the Mather et al. criteria. Although such triggers would be present at other subharmonics, only this trigger has a corresponding resonant source at (22,0). Thus, all the criteria for the subharmonic generation are met, in spite of the fact that $M_{RT}$ is less than unity.

## X. Conclusions

In this paper a case has been made for extending the conventional MPT theories to include the three-dimensional effects and the effects of the rotor-stator interactions and inflow distortions.

The necessity is created by strong and increasing experimental evidence suggesting that a modification of the existing theories is urgently needed. The most conspicuous of the experimental evidence has been the presence of subharmonic tones at subsonic relative tip speeds. The theory developed in this paper explains these phenomena and, in addition, serves as a guide for the analysis of sophisticated mode and coherence measurements.

It has been shown that the underlying mechanism, for both the shock wave description and the description of this paper, is nonlinear resonance. In the limit of rotor alone and narrow annulus, such a resonance takes place at the relative tip Mach number of unity. (This is in agreement with the prediction of the shock wave theory of MPT generation.) However, it has been shown that the nonlinear resonance can occur in two other ways if the three-dimensional and interaction effects are included.

The specific characteristics of the three kinds of MPT's are studied. A practical example from actual test data has been worked out to demonstrate one use of the theory. Other utilities would be along the direction of diagnostics and modification of the design of the fan and lining. Suggestions along those lines are included.

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## References

[1] Hawkings, D., "Multiple Tone Generation by Transonic Compressors," *Journal of Sound and Vibration,* Vol. 17, 1971, pp. 241-250.

[2] Mathews, D. C. and Nagel, R. T., "Inlet Geometry and Axial Mach Number Effects on Fan Noise Propagation," AIAA Paper 73-1022, Oct. 1973.

[3] Kazin, S.B., "Acoustic Testing of Supersonic Tip Speed Fan with Acoustic Treatment of Rotor Casing Slots (Quiet Engine Program Scale Model Fan C)," NASA CR-134501, 1973.

[4] Vaidya, P.G. and Wang, K.S., "Nonlinear Resonance as the Cause of Multiple Pure Tones," *Journal of Aircraft,* Vol. 15, 1978, pp. 526-533.

[5] Pestorius, F.M., "Propagation of Plane Acoustic Noise of Finite Amplitude," Applied Research Lab, The University of Texas, Austin, Tex., Tech. Rept. ARL-TR-73-23, 1973.

[6] Vaidya, P.G., "Alternative Approaches to the Modelling of the Multiple Pure Tones," AIAA Paper 77-1344, 1977.

[7] Kraft, R.E. and Poesy, J.W., "A Preliminary In-Duct Measurement of Spinning Modes in the Inlet of a Rotating Vehicle," 91st Meeting of the Acoustical Society of America, Washington, D.C., April 1976.

[8] Joppa, P.D., personal communication, 1983, Boeing Commercial Airplane Co., (also to be published as NASA-CR).

[9] Shivashankara, B.N., "High Bypass Ratio Engine Noise Component Separation by Coherence Technique," *Journal of Aircraft,* Vol. 20, 1983, pp. 236-242.

[10] Saule, A.V., "Some Observations about the Components of Transonic Fan Noise from Narrowband Spectral Analysis," NASA TN D-7788, Oct. 1974.

[11] Gutin, L., "On the Sound Field of a Rotating Propeller," NACA TM 1195, Oct. 1948.

[12] Tyler, J.M. and Sofrin, T.G., "Axial Flow Compressor Noise Studies," *SAE Transactions,* Vol. 70, 1962, pp. 309-322.

[13] Benzakein, M.J., "Research on Fan Noise Generation," *Journal of Acoustical Society of America,* Vol. 51, 1972, pp. 1427-1438.

[14] Mugridge, B.D. and Morfey, C.L., "Sources of Noise in Axial Flow Fans," *Journal of Acoustical Society of America,* Vol. 51, 1972, pp. 1411-1425.

[15] Merriman, J.E. and Good, R.C., "Effect of Forward Motion on Fan Noise," AIAA Paper 75-464, March 1975.

[16] Woodward, P.P. et al., "Effectiveness of an Inlet Flow Turbulence Control Device to Simulate Flight Fan Noise in an Anechoic Chamber," NASA-TM-73855, 1977.

[17]Mather, J.S.B., Savidge, J., and Fisher, M.J., "New Observations on Tone Generation in Fans," *Journal of Sound and Vibration,* Vol. 16, 1971, pp. 407-418.

[18]Morfey, C.L., "The Acoustics of Axial Flow Machines," *Journal of Sound and Vibration,* Vol. 22, 1972, pp. 445-466.

[19]Cole, J.D. and Kevorkian, J., "Uniformly Valid Asymptotic Approximations for Certain Non-linear Differential Equations," *Nonlinear Differential Equations and Nonlinear Mechanics,* Academic Press, New York, 1963, pp. 113-120.

[20]Blackstock, D.T., "Connection Between the Fay and Fubini Solutions for Plane Sound Waves of Finite Amplitude," *Journal of the Acoustical Society of America,* Vol. 39, 1966, pp. 1019-1026.

[21]Vaidya, P.G. and Wang, L. S., "Non-linear Propagation of Complex Sound Field in Rectangular Ducts, Part I," *Journal of Sound and Vibration,* Vol. 50, No. 1, 1977, pp. 29-42.

[22]Morse, P.M. and Ingard, K.V., *The Methods of Theoretical Acoustics,* McGraw-Hill Book Co., New York, 1968.

[23]Goel, N.S., Maitra, S.C., and Montroll, W.E., "On the Volterra and Other Non-linear Models of Interacting Populations," *Reviews of Modern Physics,* Vol. 43, 1971, pp. 230-276.

[24]Wang, K.S. and Vaidya, P.G., "Propagation of High Amplitude Higher Order Mode Sounds in Rigid Rectangular Ducts Carrying Mean Flow," Herrick Lab, Rept. HL-75-34, 1975.

[25]Sofrin, T.G., "Aircraft Turbomachinery Noise," *Course in the Fluid Mechanics of Turbomachinery,* Pratt and Whitney Aircraft Co., Hartford, Conn., 1973, p. 58.

[26]Vaidya, P.G., "Demonstration of Strong Resonance of Spinning Modes in Annular Ducts," 93rd Meeting of the Acoustical Society of America, Harrisburg, Pa., June 1977.

[27]Pickett, G.F., "Predictions of the Spectral Content of Combination Tone Noise," *Journal of Aircraft,* Vol. 9, Sept. 1972, pp. 658-663.